Wilcox develops a rigorous theory of approximation for ambiguity functions, based on mean square error approximation. It seems however that the theory is of little help in finding the "ideal" ambiguity function.

I. The Use of Expansions in Series of Orthonormal Functions

Given a sequence of functions $\phi_0(t), \phi_1(t), \ldots, \phi_n(t), \ldots$ which are orthonormal and complete, one can generate:

$$\psi_{mn}(x,y) = \int_{-\infty}^{\infty} \phi_m(t - x/2)\, \bar{\phi}_n(t + x/2)\, e^{-2\pi iyt}\, dt$$

The sequence $\{\psi_{mn}(x,y)\}$ is orthonormal and complete over the x-y plane.

Any function F(x,y) which is L^2 over the x-y plane can be expanded in terms of the sequence $\{\psi_{mn}(x,y)\}$. The expansion is:

$$F(x,y) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{m=0}^{N} \sum_{n=0}^{N} c_{mn}\, \psi_{mn}(x,y)$$

$$c_{mn} = \iint F(x,y)\, \bar{\psi}_{mn}(x,y)\, dx\, dy$$

II. Approximation of Arbitrary Functions by Ambiguity Functions

One wants to find the ambiguity function which best approximates a given function F(x,y) in the mean square error sense.

$$\text{minimum of} \iint |\, F(x,y) - A_u(x,y)|^2\, dx\, dy$$

F(x,y) is limited only in that

$$\iint |\, F(x,y)|^2\, dx\, dy = 1$$

It is convenient to consider the class of waveforms which can be generated from a finite set of orthonormal functions

$$u(t) = \sum_{m=0}^{N} a_m \phi_m(t)$$

$$\int \phi_m(t)\, \phi_n(t)\, dt = \delta_{mn}$$

$$\int u^2(t)\, dt = \sum_{m=0}^{N} | a_m |^2 = 1$$

Corresponding to the orthonormal set $\{\phi_m(t)\}$ there is the orthonormal set $\{\psi_{mn}(x,y)\}$. The given function F(x,y) can be expanded in terms of the $\psi_{mn}(x,y)$ as stated above, and as a result one will have a Hermitian matrix $[c]$ where

$$c_{mn} = \int\int F(x,y)\, \bar{\psi}_{mn}(x,y)\, dx\, dy$$

The solution to the approximation problem is as follows: The minimum mean square error of approximation is $2(1 - \mu_0)$ where μ_0 is the largest eigenvalue of the matrix $[c_{mn}]$. It is attained by the waveform $\sum_{m=0}^{N} a_m \phi_m(t)$ where $(a_0, a_1, \ldots, a_n)$ is any eigenvector for $[c_{mn}]$ corresponding to μ_0. The corresponding ambiguity function is:

$$A_u(x,y) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_m \bar{a}_n \psi_{mn}(x,y)$$

III. A Particular Set of Orthonormal Functions: The Hermite Waveforms

A particular set of orthonormal waveforms is the Hermite set $\{u_n(t)\}$

$$u_n(t) = \frac{2^{1/4}}{\sqrt{n!}} H_n(2\sqrt{\pi}\, t)\, e^{-\pi t^2} \qquad n = 0, 1, 2, \ldots.$$

where $$H_n(x) = (-1)^n e^{\frac{x^2}{2}} (d/dx)^n e^{-\frac{x^2}{2}} \qquad n = 0, 1, 2, \ldots.$$

Corresponding to $\{u_n(r)\}$ is the orthonormal set $\{A_{mn}(x,y)\}$

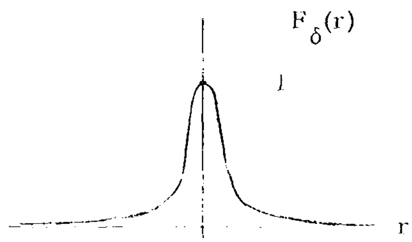
$$A_{mn}(x,y) = (-1)^{m-n}\sqrt{\frac{n!}{m!}}\; e^{-\frac{\pi}{2}(x^2+y^2)} \left(\sqrt{\pi}\,[x+iy]\right)^{m-n} L_n^{(m-n)}\left(\pi[x^2+y^2]\right)$$

where $L_n^{(\alpha)}(x) = \frac{1}{n!}\, e^x x^{-\alpha} (d/dx)^n \left(x^{n+\alpha} e^{-x}\right)$ for $x \geq 0$

or $A_{mn}(x,y) = (-1)^{m-n}\sqrt{\frac{n!}{m!}}\; e^{-\frac{\pi}{2}r^2} (\sqrt{\pi}\, r)^{m-n} e^{i\theta(m-n)} L_n^{(m-n)}(\pi r^2)$

IV. An Application of Wilcox's Theory

Let us find the ambiguity function which best approximates the "ideal" function F(r)



$$F_\delta(r) = e^{-\frac{r^2}{2\sigma^2}} + \sqrt{\frac{2}{\pi a^2}}\; e^{-\frac{r^2}{a^2}}$$

$$F(r) = \lim_{\substack{\sigma \to 0 \\ a \to \infty}} F_\delta(r)$$

Note that $F(0) = 1$

and $$\iint_S |F(r)|^2\, dS = 1$$

First find the Hermite matrix of $F_\delta(r)$:

$$C_{mn}(\sigma,a) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,y)\,\bar{\psi}_{mn}(x,y)\,dx\,dy = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} F(r,\theta)\,\bar{\psi}_{mn}(r,\theta)\,r\,d\theta\,dr$$

$$= \int_{r=0}^{\infty}\left[e^{-r^2/2\sigma^2} + \sqrt{\frac{2}{\pi a^2}}\; e^{-r^2/a^2}\right](-1)^{m-n}\sqrt{\frac{n!}{m!}}\; e^{-\frac{\pi r^2}{2}} (\sqrt{\pi}\, r)^{m-n} L_n^{(m-n)}$$

$$\bullet\; r\int_{\theta=0}^{2\pi} e^{-i\theta(m-n)}\,d\theta\,dr$$

$$= \begin{cases} 0 & \text{for } m \neq n \\ \displaystyle\int_{r=0}^{\infty}\left[e^{-r^2/2\sigma^2} + \sqrt{\frac{2}{\pi a^2}}\; e^{-r^2/a^2}\right] e^{-\frac{\pi r^2}{2}} L_n^{(0)}(\pi r^2) \bullet 2\pi\, r\, dr & \text{for } m = n \end{cases}$$

Now:

$$\int_{x=0}^{\infty} e^{-ax} L_n(x)\,dx = \int_0^{\infty} e^{-ax} \frac{1}{n!} e^{x} (d/dx)^n (x^n e^{-x})\,dx$$

$$= \int_0^{\infty} \frac{1}{n!} e^{(1-a)x} (d/dx)^n (x^n e^{-x})\,dx$$

$$= \frac{1}{n!} e^{(1-a)x} (d/dx)^{n-1} (x^n e^{-x}) \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{n!} (1-a)\, e^{(1-a)x} (d/dx)^{n-1} (x^n e^{-x})\,dx$$

The first term for large x is zero if $a > 0$. The behavior of $(d/dx)^{n-1}(x^n e^{-x})$ near the origin is as $n! \times x$, so at $x = 0$ the first term is again zero. So

$$\int_{x=0}^{\infty} e^{-ax} L_n(x)\,dx = -\int_0^{\infty} \frac{1}{n!} (1-a)\, e^{(1-a)x} (d/dx)^{n-1} (x^n e^{-x})\,dx \qquad \text{for } a > 0$$

$$= -\frac{1}{n!}(1-a)\left\{ \underbrace{\Big[e^{(1-a)x} (d/dx)^{n-2} (x^n e^{-x}) \Big]_0^{\infty}}_{0 \text{ for } a > 0} - \int_0^{\infty} (1-a)\, e^{(1-a)x} (d/dx)^{n-2} (x^n e^{-x})\,dx \right\}$$

$$= +\frac{(1-a)^2}{n!} \int_0^{\infty} e^{(1-a)x} (d/dx)^{n-2} (x^n e^{-x})\,dx$$

$$= \frac{(-1)^r (1-a)^r}{n!} \int_0^{\infty} e^{(1-a)x} (d/dx)^{n-r} (x^n e^{-x})\,dx$$

$$= \frac{(-1)^n (1-a)^n}{n!} \int_0^{\infty} e^{-ax} x^n\,dx$$

$$= \frac{(-1)^n (1-a)^n}{n!} \times \frac{n!}{a^{n+1}} = \frac{(a-1)^n}{a^{n+1}} \qquad \text{for } a > 0$$

We had

$$c_{nn}(\sigma, a) = \int_{r=0}^{\infty} \left[e^{-r^2/2\sigma^2} + \sqrt{\frac{2}{\pi a^2}}\, e^{-r^2/a^2} \right] e^{-\pi r^2/2} L_n^{(0)}(\pi r^2) \bullet 2\pi r\,dr$$

Let $\pi r^2 = x \qquad 2\pi r\,dr = dx$

$$c_{nn}(\sigma,a) = \int_{x=0}^{\infty}\left[e^{-x/2\pi\sigma^2} + \sqrt{\frac{2}{\pi a^2}}\; e^{-x/\pi a^2}\right] e^{-x/2}\, L_n(x)\,dx$$

$$= \int_{x=0}^{\infty} e^{-\left[\frac{1}{2\pi\sigma^2}+\frac{1}{2}\right]x} L_n(x)\,dx + \sqrt{\frac{2}{\pi a^2}}\int_{x=0}^{\infty} e^{-\left[\frac{1}{\pi a^2}+\frac{1}{2}\right]x} L_n(x)\,dx$$

$$= \frac{\left(\frac{1}{2\pi\sigma^2}-\frac{1}{2}\right)^n}{\left(\frac{1}{2\pi\sigma^2}+\frac{1}{2}\right)^{n+1}} + \sqrt{\frac{2}{\pi a^2}}\;\frac{\left(\frac{1}{\pi a^2}-\frac{1}{2}\right)^n}{\left(\frac{1}{\pi a^2}+\frac{1}{2}\right)^{n+1}}$$

For any value of $\sigma \neq 0$ and $a \neq \infty$, one can show that the largest $c_{nn}(\sigma,a)$ is $c_{oo}(\sigma,a)$

The matrix of coefficients of the orthonormal expansion of $F_\delta(r)$ is the diagonal matrix

$$\left[c_{mn}(\sigma,a)\right] = \begin{bmatrix} c_{00}(\sigma,a) & & & & 0\;0 \\ & c_{11}(\sigma,a) & & & 0 \\ & & c_{22}(\sigma,a) & & \\ & & & \ddots & \\ 0 & & & & \\ 0\;0 & & & & c_{nn}(\sigma,a) \end{bmatrix}$$

The largest eigenvalue of this diagonal matrix is

$$c_{oo}(\sigma,a) = \frac{1}{\frac{1}{2\pi\sigma^2}+\frac{1}{2}} + \sqrt{\frac{2}{\pi a^2}} \bullet \frac{1}{\frac{1}{\pi a^2}+\frac{1}{2}}$$

The normalized eigenvector corresponding to this eigenvalue is

$$\left. a \right] = \left. \begin{matrix} 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{matrix} \right]$$

Hence the waveform of unit energy having the ambiguity function which best approximates $F_\delta(r)$, is the zero order hermite waveform

$$u_0(t) = 2^{1/4} H_0(2\sqrt{\pi}\, t)\, e^{-\pi t^2} = 2^{1/4} e^{-\pi t^2}$$

or a gaussian pulse.

The corresponding ambiguity function is

$$A_{00}(x, y) = e^{-\frac{\pi r^2}{2}}$$

This result shows that the ambiguity function which best approximates a circularly symmetric gaussian function of arbitrary variance is the zero order Laguerre ambiguity function or the only circularly symmetric gaussian ambiguity function.

The mean square error in this particular approximation is

$$2(1 - \mu_0) = 2\,[1 - c_{oo}(\sigma, a)]$$

Now in the limit as we let $F_\delta(r)$ approach our "ideal" function $F(r)$

$$F(r) = \lim_{\substack{\sigma \to 0 \\ a \to \infty}} F_\delta(r)$$

then the largest eigenvalue in the expansion of $F(r)$ is

$$c_{oo} = \lim_{\substack{\sigma \to 0 \\ a \to \infty}} c_{oo}(\sigma, a)$$

$$= \lim_{\substack{\sigma \to 0 \\ a \to \infty}} \left[\frac{1}{\frac{1}{2\pi\sigma^2} + \frac{1}{2}} + \sqrt{\frac{2}{\pi a^2}} \cdot \frac{1}{\frac{1}{\pi a^2} + \frac{1}{2}} \right]$$

$$= 0$$

Now the error in the approximation is

$$2(1 - \mu_0) = 2(1 - c_{oo}) = 2$$

If you consider that both the "ideal" function and the approximating ambiguity function have a unity integrated square, one sees that the "best" approximation is unfortunately orthogonal to the desired delta function.

V. Conclusion and Criticism

The technique of approximating an arbitrary amplitude function by the "best" ambiguity function in the mean square sense seems to be of small immediate usefulness. The best approximation of an ideal function is no good at all.

However, the function of practical significance in the radar problem is the ambiguity function squared. It might be possible to get a good approximation to an ideal envelope function using the magnitude of the optimum ambiguity function.

An entirely different criticism of Wilcox's approach is that the mean square error criterion is not significant or correct for this approximation problem. An ideal ambiguity function includes a sharp central peak. A mean square expansion at a discontinuous point of zero measure fails entirely or converges in the mean. At the region of greatest interest, the expansion is at its worst.

A better approximation criterion would be that of Chebyshev; that is the best approximation is that which has the smallest maximum excursion from the desired function. A theory of Chebyshev approximation has not been developed over the infinite plane.

References

The information in sections 1, 2, and 3 has been taken from:

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